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A polynomial-time algorithm for the paired-domination problem on permutation graphs

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Abstract

A set S of vertices in a graph $H = (V, E)$ with no isolated vertices is a *paired-dominating set* of H if every vertex of H is adjacent to at least one vertex in S and if the subgraph induced by S contains a perfect matching. Let G be a permutation graph and π be its corresponding permutation. In this paper we present an $O(mn)$ time algorithm for finding a minimum cardinality paired-dominating set for a permutation graph G with n vertices and m edges.

Keywords: Algorithm; Permutation graph; Paired-domination

1. Introduction

In this paper we in general follow [14] for notation and graph theory terminologies. Specifically, let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E, and let v be a vertex in V. The order of G is given by $n = |V|$ and its size by $m = |E|$. The *open neighborhood* of v is defined by $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is defined by $N[v] = N(v) \cup \{v\}$. In general, let $N(S)$ and $N[S]$ denote, respectively, $\cup_{v\in S} N(v)$ and $\cup_{v\in S} N[v]$. For subsets $S, T \subseteq V$, the set S dominates the set T in G if $N[T] \subseteq N[S]$. Each vertex v of G dominates itself and every vertex adjacent to v, i.e., all vertices in its closed neighborhood. For $S \subseteq V$, let $\langle S\rangle$ denote the subgraph of G induced by S.

A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to at least a vertex in S. The *domination number* of G is the minimum cardinality of a dominating set of G. A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to a vertex of M.

A *paired-dominating set* of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph induced by S contains a perfect matching M (not necessarily induced). Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in S. Every graph without isolated vertices has a paired-dominating set since the end-vertices of any maximal matching form such a set. The *paired-domination number* of

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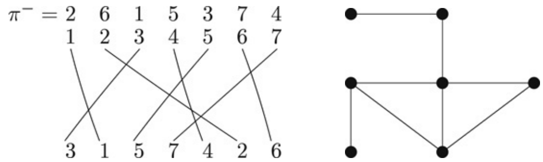

(a) The permutation diagram. (b) A permutation graph.

Fig. 1. A permutation graph and its permutation diagram.

G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired-dominating set. The minimum paired-dominating set problem, abbreviated as MPDS, is to find a paired-dominating set S of G such that $|S|$ is minimized. Paired-domination was introduced by Haynes and Slater [14] as a model for assigning backups to guards for security purposes, and has been studied from the theoretic point of view, for example, in [2–4,7,8,10,11,15–19,21,25–27,29], among others.

The aim of this paper is to investigate the problem of determining $\gamma_{pr}(G)$ for a permutation graph G from the algorithmic point of view. The decision problem to determine a minimum cardinality paired-dominating set of an arbitrary graph has been known to be NP-complete (see [13]). For the special case of trees, Qiao et al. [26] presented a linear time algorithm. Cheng et al. [8] proposed an $O(m+n)$ and $O(m(m+n))$ time algorithms to solve the MPDS problem for interval graphs and circular-arc graphs, respectively. The literature on algorithmic aspects of domination in graphs has been by surveyed and detailed by Chang [5].

Let $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ be a permutation on the set $V_n = \{1, 2, \ldots, n\}$. Then the *permutation graph* $G[\pi] = (V, E)$ is the undirected graph such that $V = V_n$ and $(i, j) \in E$ if and only if

$$(i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0,$$

where $\pi^{-1}(i)$ is the position of i in $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$. Throughout the paper, we assume that the input is a permutation $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$, and the given permutation graph G contains no isolated vertices.

A permutation graph is an intersection graph based upon the *permutation diagram* Fig. 1, which is defined as follows: Write the number $1, 2, \ldots, n$ horizontally from left to right. Under every i, write the number $\pi(i)$. Draw line segments connecting i in the top row and i in the bottom row, for each i. It is easy to see that two vertices i and j of $G[\pi]$ are adjacent if and only if the corresponding line segments of i and j intersect. Fig. 1 shows the permutation graph $G[\pi]$ where its corresponding permutation diagram of a permutation $\pi[3, 1, 5, 7, 4, 2, 6]$. The permutation graphs are known to have a variety of practical applications [12,24] and for this reason, many algorithms for determining parameters in graph theory have been developed in the literature [1,6,9,20,22,23,28,30–32].

In this paper, we propose an efficient $O(mn)$ algorithm for solving the MPDS problem on permutation graphs. Our algorithm is based on a recursive formula by using the dynamic programming method. In Section 2, we describe our recursive formula of the dynamic programming. Our algorithm is described in Section 3. Section 4 contains some conclusions.

2. A dynamic programming approach

In this section we shall describe our basic approach based upon the dynamic programming approach. Essentially, we want to find an MPDS of $\{\pi_1, \pi_2, \ldots, \pi_n\}$ dominating $\{1, 2, \ldots, n\}$. In the following, we may assume that the permutation graph $G[\pi]$ discussed below is connected; otherwise we look at each (connected) component separately.

For convenience, we introduce more notation as follows:

(1) For any $1 \le i, j \le n$, and $V_i = \{\pi_1, \pi_2, \ldots, \pi_i\}$, denote $V_{i,j}$ as the subset of V_i containing all elements smaller than or equal to j, i.e., $V_{i,j} = \{\pi_k \in V_i \mid \pi_k \le j\}$. Clearly, $V_{i,j} \subseteq V_i$.

(2) For each i, $1 \le i \le n$, denote π_i^* as the minimum number over the suffix $\pi_i, \pi_{i+1}, \ldots, \pi_n$, i.e., $\pi_i^* = \min\{\pi_i, \pi_{i+1}, \ldots, \pi_n\}$, and set $V_i^* = V_i \cup \{\pi_i^*\}$.

(3) For any vertex set S, define $\max(S)$ as the maximum number in S.

(4) For a family $\mathcal{F}$ of sets of vertices, $\mathrm{Min}(\mathcal{F})$ denotes a minimum cardinality set S in $\mathcal{F}$ and $\max(S)$ is as large as possible if $\mathcal{F}$ is not the empty set; $\mathrm{Min}(\mathcal{F})$ denotes a set of infinite cardinality otherwise. $\mathrm{Min}(\mathcal{F})$ may not be unique. If there is more than one candidate for $\mathrm{Min}(\mathcal{F})$, we select arbitrarily one of the candidates.

Lemma 1. *For a permutation graph $G[\pi]$ with no isolated vertices, $\langle V_i^* \rangle$ has no isolated vertices for each i, $1 \le i \le n$.*

Proof. Suppose to the contrary that there exists an i_0 ($1 \le i_0 \le n$) such that $\langle V_{i_0}^* \rangle$ has an isolated vertex π_l ($l \le i_0$). Then $\pi_l \le \pi_{i_0}^*$, for otherwise $(\pi_l, \pi_{i_0}^*) \in E(G)$. If $\pi_l = \pi_{i_0}^*$ ($= \min\{\pi_{i_0}, \pi_{i_0+1}, \dots, \pi_n\}$), then $\pi_l = \pi_{i_0}$. Hence, π_{i_0} is an isolated vertex in G, contradicting the assumption of the lemma. If $\pi_l < \pi_{i_0}^*$, then $\pi_l = l$. Thus, for $1 \le i < l$, $\pi_i < l$, and for $l < i \le n$, $\pi_i > l$. This implies that π_l is an isolated vertex in G, contradicting our assumption again. □

By Lemma 1, we see that $\langle V_i^* \rangle$ has no isolated vertices, so it is clear that for each i and j, $1 \le i, j \le n$, there exists a subset D of V_i^* such that D dominates all the vertices of $V_{i,j}$ and $\langle D \rangle$ has a perfect matching in $\langle V_i^* \rangle$.

Based on Lemma 1, for each i and j, $1 \le i, j \le n$, we define $PD_{i,j}$ as follows:

(i) $PD_{i,j}$ is a minimum cardinality subset S of V_i^* such that S is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_i^* \rangle$;
(ii) $\max(PD_{i,j})$ is as large as possible.

In particular, we define $PD_{0,j} = \emptyset$ for $1 \le j \le n$. Clearly, $PD_{n,n}$ is a desired minimum cardinality paired-dominating set for $G[\pi]$.

We define $X = \{S : S \subseteq V_i^*$ such that S is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_i^* \rangle\}$, and we further partition X into three subsets: $X_1 = \{S \in X : \pi_i^* \in S\}$, $X_2 = \{S \in X : \pi_i^* \notin S, \pi_i \in S\}$ and $X_3 = \{S \in X : \pi_i^* \notin S, \pi_i \notin S\}$.

Following the above definitions, we have

$$PD_{i,j} = \begin{cases} \emptyset & \text{if } V_{i,j} = \emptyset, \\ \mathrm{Min}(X) & \text{otherwise.} \end{cases}$$

Consider the case $i = 1$. If $j < \pi_1$, then $V_{1,j} = \{\pi_1\} \cap \{1, 2, \dots, j\} = \emptyset$, and so $PD_{1,j} = \emptyset$. Otherwise, $V_{1,j} = \{\pi_1\}$. According to our assumption that G contains no isolated vertices, we have $\pi_1 \ne 1$. Then $\pi_1^* = 1$ and $V_1^* = \{1, \pi_1\}$. Hence $PD_{1,j} = \{1, \pi_1\}$. So we obtain

$$PD_{1,j} = \begin{cases} \emptyset & \text{if } j < \pi_1, \\ \{1, \pi_1\} & \text{otherwise.} \end{cases}$$

We first give several basic lemmas that will be useful for the proof of our recursive formula $PD_{i,j}$.

Lemma 2 (*Chao et al. [6]*). *For positive integers i_1, i_2 and j, if $1 \le i_1 < i_2 \le n$ and $1 \le j \le n$, then $V_{i_1,j} \subseteq V_{i_2,j}$ and $V_{i_1}^* \subset V_{i_2}^*$.*

Lemma 3. *For $1 \le i < j < k \le n$ and $\pi_k < \pi_j < \pi_i$, if w is adjacent to π_j, then w is adjacent to at least one of π_k and π_i.*

Proof. The proof is straightforward and omitted. □

Lemma 4. *For $1 < l \le i$, there exists a PD_{l-1,π_i^*} such that $\pi_i^* \notin PD_{l-1,\pi_i^*}$.*

Proof. Let S be a PD_{l-1,π_i^*}. Thus $S \subseteq V_{l-1}^*$ is a dominating set of $\langle V_{l-1,\pi_i^*} \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_{l-1}^* \rangle$. If $\pi_i^* \notin S$, then the desired result follows. If $\pi_i^* \in S$, then $\pi_i^* = \pi_{l-1}^*$ as $S \subseteq V_{l-1}^*$. Hence, there exists a vertex $\pi_{i'} \in S$ ($i' \le l-1$) such that $\pi_i^*, \pi_{i'}$ are paired in S. So, we have $\pi^{-1}(\pi_i^*) > i'$ and $(\pi^{-1}(\pi_i^*) - i')(\pi_i^* - \pi_{i'}) < 0$. Thus $\pi_{i'} > \pi_i^*$. We claim that $N(\pi_{i'}) \cap V_{l-1}^* - S \ne \emptyset$. If this is not so, then $\pi_{i'}$ dominates no vertices of V_{l-1,π_i^*}, and so does π_i^* as $\pi_{i'} > \pi_i^*$. This means that $S - \{\pi_{i'}, \pi_i^*\}$ ($\subseteq V_{l-1}^*$) is a dominating set of $\langle V_{l-1,\pi_i^*} \rangle$ and $\langle S - \{\pi_{i'}, \pi_i^*\} \rangle$ has a perfect matching in $\langle V_{l-1}^* \rangle$. Thus $S - \{\pi_{i'}, \pi_i^*\}$ is a PD_{l-1,π_i^*}, which contradicts the minimality of S. Let $\pi_{i''} \in N(\pi_{i'}) \cap V_{l-1}^* - S$ and $S' = S \cup \{\pi_{i''}\} - \{\pi_i^*\}$. Then S' ($\subseteq V_{l-1}^*$) is a dominating set of $\langle V_{l-1,\pi_i^*} \rangle$ and $\langle S' \rangle$ has a perfect matching in $\langle V_{l-1}^* \rangle$ with $|S'| = |S|$ and $\max(S') \ge \max(S)$. So S' is a PD_{l-1,π_i^*}, satisfying $\pi_i^* \notin S'$, as required. □

For $1 < i \le n$, we define

$$PD_{\pi_i^*} = \mathrm{Min}(\{PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} : \pi_l \in N(\pi_i^*), \pi_i^* \notin PD_{l-1,\pi_i^*}, l \le i\})$$

and

$$PD_{\max} = \begin{cases} PD_{i-1,j} \cup \{\pi_i, \max(V_i)\} & \text{if } \pi_i \neq \max(V_i), \\ V_i & \text{otherwise.} \end{cases}$$

By Lemma 4, $PD_{\pi_i^*} \neq \emptyset$. The following Lemmas 5 and 6 assert that $PD_{\pi_i^*}$ and $PD_{\max}$ (if $\max(V_i) \neq \pi_i$ and $\max(PD_{i-1,j}) < \pi_i$) are candidates for computing $PD_{i,j}$.

Lemma 5. *For any integers i and j, $1 < i \le n$ and $1 \le j \le n$, $PD_{\pi_i^*} \in X_1 (\subseteq X)$.*

Proof. By the definition of $PD_{\pi_i^*}$, $\pi_i^* \notin PD_{l-1,\pi_i^*}$, while PD_{l-1,π_i^*} is a minimum dominating set of $\langle V_{l-1,\pi_i^*}\rangle$. We claim $\pi_l \notin PD_{l-1,\pi_i^*}$. If this is not the case, then it is easy to see that $\pi_l = \pi_{l-1}^* \le \pi_i^*$. On the other hand, since $\pi_l \in N(\pi_i^*)$ $(l \le i)$, $\pi_l > \pi_i^*$, which is impossible. From Lemma 2, $V_{l-1}^* \subseteq V_i^*$ as $l \le i$. Hence, $PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} \subseteq V_i^*$. We next show that each vertex of $V_{i,j} - V_{l-1,\pi_i^*}$ is dominated by π^* or π_l. Let $\pi_k \in V_{i,j} - V_{l-1,\pi_i^*}$. If $\pi_k > \pi_i^*$, then $(\pi_k - \pi_i^*)(k - \pi^{-1}(\pi_i^*)) < 0$, and so $(\pi_k, \pi_i^*) \in E$. If $\pi_k < \pi_i^*$, then $k \ge l$. Since $\pi_l \in N(\pi_i^*)$ and $l \le i$, $\pi_l > \pi_i^*$, then $\pi_l > \pi_i^* > \pi_k$. This implies that $(\pi_k - \pi_l)(k - l) \le 0$, i.e., $\pi_k = \pi_l$ or $(\pi_k, \pi_l) \in E$. Hence, all the vertices in $V_{i,j}$ are dominated by $PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\}$. Therefore, $PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} \in X_1$. Note that $PD_{\pi_i^*} = \mathrm{Min}(\{PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} : \pi_l \in N(\pi_i^*), l \le i\})$, so $PD_{\pi_i^*} \in X_1$, as desired. $\square$

Lemma 6. *For any integers i and j, $1 < i \le n$ and $1 \le j \le n$, if $\max(V_i) \neq \pi_i$ and $\max(PD_{i-1,j}) < \pi_i$, then $PD_{\max} \in X$.*

Proof. Clearly, $PD_{\max} \subseteq V_i^*$. Since $\max(V_i) \neq \pi_i$ and $\max(PD_{i-1,j}) < \pi_i$, $\pi_i \notin PD_{i-1,j}$ and $\pi_i < \max(V_i)$, and thus $\max(V_i) \notin PD_{i-1,j}$ and $(\max(V_i), \pi_i) \in E$. Note that $V_{i,j} - V_{i-1,j} \subseteq \{\pi_i\}$, and we have $PD_{\max} = PD_{i-1,j} \cup \{\pi_i, \max(V_i)\}$ as a dominating set of $\langle V_{i,j}\rangle$ and $\langle PD_{\max}\rangle$ has a perfect matching in $\langle V_i^*\rangle$, the desired result follows. $\square$

In order to present the recursive formula of $PD_{i,j}$ for the case of $1 < i \le n$, we further prove the following several lemmas.

Lemma 7. *For each $S \in \mathrm{Min}(X_1)$, let $\pi_l = \max(S)$. Then $\pi_i^* < \pi_l$ and $\pi_l \in N(\pi_i^*)$.*

Proof. By the definition of X_1, we have $\pi_i^* \in S$. Suppose $\pi_i^* \ge \pi_l$, then $\max(S) = \pi_i^*$. This implies that π_i^* is an isolated vertex of $\langle S\rangle$, which contradicts the assumption that $\langle S\rangle$ has a perfect matching in $\langle V_i^*\rangle$. So $\pi_i^* < \pi_l$. Furthermore, since $(\pi_l - \pi_i^*)(l - \pi^{-1}(\pi_i^*)) < 0$, $(\pi_i^*, \pi_l) \in E$, and thus $\pi_l \in N(\pi_i^*)$. $\square$

By the definition of $\mathrm{Min}(X_1)$, all the candidates S for $\mathrm{Min}(X_1)$ have the same $\max(S)$. Let $S \in \mathrm{Min}(X_1)$, $\pi_l = \max(S)$ and let M be a perfect matching in $\langle S\rangle$.

Lemma 8. *For any integers i and j, $1 < i \le n$ and $1 \le j \le n$, if there exist π_{i_1} $(i_1 < l)$ and $\pi_{l'}$ such that $(\pi_i^*, \pi_{i_1}) \in M$ and $(\pi_l, \pi_{l'}) \in M$, then $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.*

Proof. By Lemma 5, it suffices to show that there exists an $S^* \in PD_{\pi_i^*} \cap X_1$ such that $\max(S^*) \ge \max(S) = \pi_l$. Note that $\max(S) = \pi_l > \pi_{l'} \in S$ and $(\pi_l, \pi_{l'}) \in M$, so $l' > l$. We distinguish the following two cases depending on whether or not π_{l-1}^* is equal to π_i^*.

Case 1. Suppose first $\pi_{l-1}^* = \pi_i^*$. In this case, we claim that $N(\pi_{i_1}) \cap V_l - S \neq \emptyset$. Otherwise, since $\pi_i^* < \pi_{l'} < \pi_l$ and $l < l' < \pi^{-1}(\pi_i^*)$, by Lemma 3, each vertex dominated by $\pi_{l'}$ in G is adjacent to π_l or π_i^*. Furthermore, for each $t > l$, $\pi_t \in V_{i,j}$, it is dominated by π_i^* as $\pi_t > \pi_i^*$ $(= \pi_{l-1}^*)$. This implies that $S - \{\pi_{i_1}, \pi_{l'}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S - \{\pi_{i_1}, \pi_{l'}\}\rangle$ has a perfect matching $M \cup \{(\pi_i^*, \pi_l)\} - \{(\pi_i^*, \pi_{i_1}), (\pi_l, \pi_{l'})\}$ in $\langle V_i^*\rangle$ by making a pair of π_l and π_i^*, contradicting the minimality of S. Let $\pi_{i_1'} \in N(\pi_{i_1}) \cap V_l - S$ and let $S_1 = S \cup \{\pi_{i_1'}\} - \{\pi_{l'}\}$. Then $S_1 \subseteq V_i^*$ is a dominating set of $\langle V_{i,j}\rangle$ and $M_1 = (M \cup \{(\pi_{i_1'}, \pi_{i_1}), (\pi_l, \pi_i^*)\}) - \{(\pi_i^*, \pi_{i_1}), (\pi_l, \pi_{l'})\}$ is a perfect matching in $\langle S_1\rangle$. So $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \ge \max(S)$ such that $\pi_{l'} \notin S_1$ and $\pi_{l-1}^* \in S_1$.

For any $\pi_k \in S_1$, where $l < k \le i$, there exists $\pi_{k'}$ such that $(\pi_k, \pi_{k'}) \in M_1$. We claim that $k' < l$ and $N(\pi_{k'}) \cap V_l - S_1 \ne \emptyset$. Indeed, if $k' > l$, then for each vertex $\pi_t \in N(\{\pi_k, \pi_{k'}\}) \cap V_l - S$, we have $\pi_t > \pi_k > \pi^*_{l-1} = \pi^*_i$ or $\pi_t > \pi_{k'} > \pi^*_{l-1} = \pi^*_i$, so π_t is dominated by π^*_i. Moreover, note that for each vertex $\pi_t \in V_{i,j}$, $l < t \le i$, it is also dominated by π^*_i as $\pi_t \ge \pi^*_i$ $(= \pi^*_{l-1})$. This implies that $S_1 - \{\pi_k, \pi_{k'}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S_1 - \{\pi_k, \pi_{k'}\}\rangle$ still has a perfect matching in $\langle V^*_i\rangle$, which contradicts the minimality of S_1. So $k' < l$. We further show that $N(\pi_{k'}) \cap V_l - S_1 \ne \emptyset$. Otherwise, since $k' < l < k$ and $(\pi_k, \pi_{k'}) \in E$, $\pi_{k'} > \pi_k > \pi^*_{l-1} = \pi^*_i$, then $\pi_{k'}$ is dominated by π^*_i. As above, we deduce that $S_1 - \{\pi_k, \pi_{k'}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S_1 - \{\pi_k, \pi_{k'}\}\rangle$ has a perfect matching in $\langle V^*_i\rangle$, a contradiction. Let $\pi_{k''} \in N(\pi_{k'}) \cap V_l - S_1$ and let $S_2 = S_1 \cup \{\pi_{k''}\} - \{\pi_k\}$. Then $S_2 \subseteq V^*_i$ is a dominating set of $\langle V_{i,j}\rangle$ with $|S_2| = |S_1|$ and $\langle S_2\rangle$ has a perfect matching in $\langle V^*_i\rangle$ and $\max(S_2) \ge \max(S_1)$. For any $\pi_s \in S_2$, where $l < s \le i$, continuing the process as above, we can obtain after a finite number of steps a set $S^* \subseteq V^*_i$ satisfying the following conditions:

(i) $S^* \cap (\{\pi_{l+1}, \pi_{l+2}, \dots, \pi_i\} - \{\pi^*_i\}) = \emptyset$;
(ii) $S^* \subseteq V^*_i$ is a dominating set of $\langle V_{i,j}\rangle$ with $|S^*| = |S|$ and $\langle S^*\rangle$ in $\langle V^*_i\rangle$ has a perfect matching in which π^*_i and π_l are paired;
(iii) $\max(S^*) \ge \max(S)$.

Then $S^* \in X_1$. Since $\pi^*_i < \pi_l$, it follows that no vertex in V_{l-1,π^*_i} is dominated by π^*_i or π_l, so $S^* - \{\pi^*_i, \pi_l\}$ is a dominating set of $\langle V_{l-1,\pi^*_i}\rangle$ and $\langle S^* - \{\pi^*_i, \pi_l\}\rangle$ in $\langle V^*_{l-1}\rangle$ has a perfect matching. By the minimality of S^*, we deduce that $S^* - \{\pi^*_i, \pi_l\} \subseteq V^*_{l-1}$ is a minimum cardinality dominating set of $\langle V_{l-1,\pi^*_i}\rangle$ and contains a perfect matching. Then $S^* - \{\pi^*_i, \pi_l\}$ is a PD_{l-1,π^*_i}, and thus S^* is a $PD_{\pi^*_i}$. Hence, $|S| = |S^*| = |PD_{l-1,\pi^*_i}| + 2$. Note that $|PD_{\pi^*_i}| \le |PD_{l-1,\pi^*_i}| + 2 = |S|$ and if $|PD_{\pi^*_i}| = |PD_{l-1,\pi^*_i}| + 2$, then $\max(PD_{\pi^*_i}) = \max(S^*) \ge \max(S)$. So $\mathrm{Min}(X_1 \cup \{PD_{\pi^*_i}\}) = PD_{\pi^*_i}$.

Case 2. Suppose $\pi^*_{l-1} \ne \pi^*_i$. As in Case 1, we first find a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \ge \max(S)$ such that $\pi_{l'} \notin S_1$ and $\pi^*_{l-1} \in S_1$.

Suppose $\pi^*_{l-1} \notin S$. Since $\pi^*_{l-1} < \pi^*_i < \pi_{i_1}$, $(\pi^{-1}(\pi_{i_1}) - \pi^{-1}(\pi^*_{l-1}))(\pi_{i_1} - \pi^*_{l-1}) < 0$, then $(\pi_{i_1}, \pi^*_{l-1}) \in E$. Let $S_1 = S \cup \{\pi^*_{l-1}\} - \{\pi_{l'}\}$. Clearly, $S_1 \subseteq V^*_i$. We further show that S_1 is a dominating set of $\langle V_{i,j}\rangle$. It suffices to show that all the vertices dominated by $\pi_{l'}$ can be dominated by S_1. Indeed, let $\pi_t \in N(\pi_{l'})$. If $t > l$, it follows from $\pi_l > \pi^*_i$ that $\pi_t < \pi_l$ or $\pi_t > \pi^*_i$. Observe that $\pi_{l'} < \pi_l$ and $l < l' \le i \le \pi^-(\pi^*_i)$, then π_t is dominated by π_l or π^*_i. If $t < l$ $(< l')$, then $\pi_t > \pi_{l'} \ge \pi^*_{l-1}$, and so π_t is dominated by π^*_{l-1}. Therefore, S_1 is a dominating set of $\langle V_{i,j}\rangle$ and $M_1 = M \cup \{(\pi_{i_1}, \pi^*_{l-1}), (\pi_l, \pi^*_i)\} - \{(\pi^*_i, \pi_{i_1}), (\pi_l, \pi_{l'})\}$ is a perfect matching in $\langle S_1\rangle$. So $S_1 \in X_1$ and $\max(S_1) = \max(S)$ such that $\pi_{l'} \notin S_1$ and $\pi^*_{l-1} \in S_1$.

Suppose $\pi^*_{l-1} \in S$. Let $(\pi^*_{l-1}, \pi_{l_1}) \in M$. We claim that $N(\pi_{l_1}) \cap V_l - S \ne \emptyset$. If this is not so, then, for each vertex $\pi_t \in N(\pi_{l_1}) - S$, $l < t \le i$. This implies that $\pi_t < \pi_l$ or $\pi_t > \pi_l > \pi^*_i$, and thus it is dominated by π_l or π^*_i. On the other hand, note that all the vertices dominated by $\pi_{l'}$ can be dominated by π^*_i or π_l as above. So $S - \{\pi_{l'}, \pi_{l_1}\}$ is a dominating set of $\langle V_{i,j}\rangle$. Further, since $\pi_{i_1} > \pi^*_i > \pi^*_{l-1}$, $(\pi^*_{l-1}, \pi_{i_1}) \in E$, then $\langle S - \{\pi_{l'}, \pi_{l_1}\}\rangle$ has a perfect matching in $\langle V^*_i\rangle$ by making pairs of π_l and π^*_i, π^*_{l-1} and π_{i_1}, which contradicts the minimality of S. Let $\pi_{l'_1} \in N(\pi_{l_1}) \cap V_l - S$ and let $S_1 = S \cup \{\pi_{l'_1}\} - \{\pi_{l'}\}$. Then S_1 is a dominating set of $\langle V_{i,j}\rangle$ and $M_1 = M \cup \{(\pi_{l_1}, \pi_{l'_1}), (\pi_l, \pi^*_i), (\pi_{i_1}, \pi^*_{l-1})\} - \{(\pi^*_i, \pi_{i_1}), (\pi_l, \pi_{l'}), (\pi_{l-1}, \pi_{l_1})\}$ is a perfect matching in $\langle S_1\rangle$. So $S_1 \in X$ and $\max(S_1) \ge \max(S)$ such that $\pi_{l'} \notin S_1$ and $\pi^*_{l-1} \in S_1$.

For any $\pi_k \ne \pi^*_{l-1}$, $\pi_k \in S_1$, where $l < k \le i$, there exists a $\pi_{k'} \in S_1$ such that $(\pi_k, \pi_{k'}) \in M_1$. We claim that $k' < l$ and $N(\pi_{k'}) \cap V_l - S_1 \ne \emptyset$. In fact, if $k' > l$, then for each vertex $\pi_t \in N(\{\pi_k, \pi_{k'}\}) \cap V_l - S$, we have $\pi_t > \pi_k > \pi^*_{l-1}$ or $\pi_t > \pi_{k'} > \pi^*_{l-1}$, so π_t is dominated by π^*_{l-1}. Moreover, for each vertex $\pi_t \in V_{i,j}$, $l < t \le i$, we have $\pi_t < \pi_l$ or $\pi_t > \pi_l > \pi^*_i$, so π_t is dominated by π^*_i or π_l. This implies that $S_1 - \{\pi_k, \pi_{k'}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S_1 - \{\pi_k, \pi_{k'}\}\rangle$ still has a perfect matching in $\langle V^*_i\rangle$, which contradicts the minimality of S_1. So $k' < l$. Similar to the discussion in Case 1, we can deduce that $N(\pi_{k'}) \cap V_l - S_1 \ne \emptyset$.

Let $\pi_{k''} \in N(\pi_{k'}) \cap V_l - S'$ and let $S_2 = S_1 \cup \{\pi_{k''}\} - \{\pi_k\}$. Then $S_2 \subseteq V^*_i$ is a dominating set of $\langle V_{i,j}\rangle$ with $|S_2| = |S_1|$ and $\langle S_2\rangle$ has a perfect matching in $\langle V^*_i\rangle$ and $\max(S_2) \ge \max(S_1)$. Proceeding as above, we get a set $S^* \subseteq V^*_i$ satisfying the following conditions:

(i) $S^* \cap (\{\pi_{l+1}, \pi_{l+2}, \dots, \pi_i\} - \{\pi^*_i\}) = \pi^*_{l-1}$;
(ii) S^* is a dominating set of $\langle V_{i,j}\rangle$ with $|S^*| = |S|$ and $\langle S^*\rangle$ in $\langle V^*_i\rangle$ has a perfect matching in which π^*_i and π_l are paired;
(iii) $\max(S^*) \ge \max(S)$.

Then $S^* \in X_1$. As in Case 1, it can be verified that no vertex in V_{l-1,π_i^*} is dominated by π_i^* or π_l since $\pi_i^* < \pi_l$, so $S^* - \{\pi_i^*, \pi_l\}$ is a dominating set of $\langle V_{l-1,\pi_i^*}\rangle$ and $\langle S^* - \{\pi_i^*, \pi_l\}\rangle$ in $\langle V_{l-1}^*\rangle$ has a perfect matching. By the minimality of S^*, it follows that $S^* - \{\pi_i^*, \pi_l\} \subseteq V_{l-1}^*$ is a minimum cardinality dominating set of $\langle V_{l-1,\pi_i^*}\rangle$. Then $S^* - \{\pi_i^*, \pi_l\}$ is a PD_{l-1,π_i^*}, and thus S^* is a $PD_{\pi_i^*}$. Hence, $|S| = |S^*| = |PD_{l-1,\pi_i^*}| + 2$. Note that $|PD_{\pi_i^*}| \le |PD_{l-1,\pi_i^*}| + 2 = |S|$ and if $|PD_{\pi_i^*}| = |PD_{l-1,\pi_i^*}| + 2$, then $\max(PD_{\pi_i^*}) = \max(S^*) \ge \max(S)$. Therefore, $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$. □

Lemma 9. *For any integers i and j, $1 < i \le n$ and $1 \le j \le n$, if there exist π_{i_1} $(i_1 > l)$ and $\pi_{l'}$ such that $(\pi_i^*, \pi_{i_1}) \in M$ and $(\pi_l, \pi_{l'}) \in M$, then* $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

Proof. Similar to Lemma 8, we need to show that there exists an $S^* \in PD_{\pi_i^*} \cap X_1$ such that $\max(S^*) \ge \max(S)$. We claim that $\pi_{l-1}^* \ne \pi_i^*$, $\pi_{l-1}^* \notin S$, and $N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} \ne \emptyset$. We first show that $\pi_{l-1}^* \ne \pi_i^*$. Suppose to the contrary that $\pi_{l-1}^* = \pi_i^*$, then it is easy to see that $\pi_i^* < \pi_{l'} < \pi_l$ and $\pi_i^* < \pi_{i_1} < \pi_l$. Hence, by Lemma 3, $S - \{\pi_{l'}, \pi_{i_1}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S - \{\pi_{l'}, \pi_{i_1}\}\rangle$ has a perfect matching in $\langle V_i^*\rangle$ by pairing π_i^* with π_l, which contradicts the minimality of S. So $\pi_{l-1}^* \ne \pi_i^*$. Second, we show that $\pi_{l-1}^* \notin S$. Suppose this is not the case, $\pi_{l-1}^* \in S$. For any vertex $\pi_t \in N[\pi_{i_1}]$, if $t < i_1$, then $\pi_t > \pi_{i_1}$. By our assumption that $(\pi_i^*, \pi_{i_1}) \in M$, we have $\pi_{i_1} > \pi_i^*$ as $i_1 < \pi^-(\pi_i^*)$. Hence, $(\pi_t, \pi_i^*) \in E$. If $t \ge i_1$ $(> l)$, then $\pi_t \le \pi_{i_1} < \pi_l$, and thus $(\pi_t, \pi_l) \in E$. So $N[\pi_{i_1}] \subseteq N[\pi_l] \cup N[\pi_i^*]$. For any vertex $\pi_t \in N[\pi_{l'}]$, if $t \le l-1$, then $\pi_t > \pi_{l'} \ge \pi_{l-1}^*$ and $t \le l-1 \le \pi^-(\pi_{l-1}^*)$, so $(\pi_t, \pi_{l-1}^*) \in E$. If $l < t < l'$, then $\pi_t < \pi_l$ or $\pi_t > \pi_l > \pi_i^*$ and $l' \le \pi^-(\pi_i^*)$, and thus $(\pi_t, \pi_l) \in E$ or $(\pi_t, \pi_i^*) \in E$. If $t \ge l'$ $(> l)$, then $\pi_l > \pi_{l'} \ge \pi_t$, so $(\pi_t, \pi_l) \in E$. So $N[\pi_{l'}] \subseteq N[\pi_l] \cup N[\pi_{l-1}^*] \cup N[\pi_i^*]$. Let $S' = S - \{\pi_{l'}, \pi_{i_1}\}$. Then S' is a dominating set of $\langle V_{i,j}\rangle$ and $M' = M \cup \{(\pi_l, \pi_i^*)\} - \{(\pi_l, \pi_{l'}), (\pi_i^*, \pi_{i_1})\}$ is a perfect matching in $\langle S'\rangle$. This contradicts the minimality of S. So $\pi_{l-1}^* \notin S$. Finally, we show that $N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} \ne \emptyset$. If $N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} = \emptyset$, then $N(\pi_{l'}) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} = \emptyset$, so we have $N[\pi_{l'}] \subseteq N[\pi_l] \cup N[\pi_i^*]$. Hence, $S - \{\pi_{l'}, \pi_{i_1}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S - \{\pi_{l'}, \pi_{i_1}\}\rangle$ has a perfect matching in $\langle V_i^*\rangle$, contradicting the minimality of S.

Let $\pi_{l_1} \in N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\}$ and $S_1 = S \cup \{\pi_{l-1}^*, \pi_{l_1}\} - \{\pi_{l'}, \pi_{i_1}\}$. Since $N[\pi_{i_1}] \subseteq N[\pi_l] \cup N[\pi_i^*]$ and $N[\pi_{l'}] \subseteq N[\pi_l] \cup N[\pi_{l-1}^*] \cup N[\pi_i^*]$, S_1 is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S_1\rangle$ has a perfect matching in $\langle V_i^*\rangle$ by pairing $\{\pi_l, \pi_i^*\}$ and $\{\pi_{l-1}^*, \pi_{l_1}\}$. So $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \ge \max(S)$ such that $\pi_{l'} \notin S_1$ and $\pi_{l-1}^* \in S_1$. Using analogous arguments as in Lemma 8, we can get a set $S^* \in X_1$ such that $S^* - \{\pi_i^*, \pi_l\}$ is a PD_{l-1,π_i^*} and S^* is a $PD_{\pi_i^*}$. Hence, $|S| = |S^*| = |PD_{l-1,\pi_i^*}| + 2$. Note that $|PD_{\pi_i^*}| \le |PD_{l-1,\pi_i^*}| + 2 = |S|$ and if $|PD_{\pi_i^*}| = |PD_{l-1,\pi_i^*}| + 2$, then $\max(PD_{\pi_i^*}) = \max(S^*) \ge \max(S)$. Therefore, $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$. □

Lemma 10. *For any integers i and j, $1 < i \le n$ and $1 \le j \le n$, if $(\pi_i^*, \pi_l) \in M$, then* $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

Proof. Similar to Lemma 8, we again need to show that there exits an $S^* \in PD_{\pi_i^*} \cap X_1$ such that $\max(S^*) \ge \max(S)$. We consider the following two cases depending on whether or not π_{l-1}^* is equal to π_i^*.

Case 1. Suppose $\pi_{l-1}^* = \pi_i^*$. Then, for any $\pi_k \in S$ for $l < k < i$, there exists $\pi_{k'} \in S$ such that $(\pi_k, \pi_{k'}) \in M$. Similar to the discussion for S_1 in Case 1 of Lemma 8, we can obtain a set $S^* \in X_1$ satisfying the conditions (i)–(iii) in Case 1 of Lemma 8 and S^* is a $PD_{\pi_i^*}$ with $\max(PD_{\pi_i^*}) \ge \max(S)$. Therefore, $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

Case 2. Suppose $\pi_{l-1}^* \ne \pi_i^*$. If $\pi_{l-1}^* \in S$, then we deal with S as in Case 2 of Lemma 8 for S_1. Finally, we can obtain a set $S^* \in X_1$ satisfying the conditions (i)–(iii) in Case 2 of Lemma 8 and S^* is a $PD_{\pi_i^*}$ with $\max(PD_{\pi_i^*}) \ge \max(S)$. Hence, $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$, thus the assertion holds. In what follows, we may assume that $\pi_{l-1}^* \notin S$. As in Case 1 of Lemma 8, we first find a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \ge \max(S)$ such that $\pi_{l-1}^* \in S_1$.

Suppose $S \cap (\{\pi_{l+1}, \ldots, \pi_i\} - \{\pi_i^*\}) = \emptyset$. Since $\pi_i^* < \pi_l$, it follows that no vertex in V_{l-1,π_i^*} is dominated by π_i^* or π_l, so $S - \{\pi_i^*, \pi_l\}$ is a dominating set of $\langle V_{l-1,\pi_i^*}\rangle$ and $\langle S - \{\pi_i^*, \pi_l\}\rangle$ in $\langle V_{l-1}^*\rangle$ has a perfect matching. By minimality of S, we deduce that $S - \{\pi_i^*, \pi_l\} \subseteq V_{l-1}^*$ is a minimum cardinality dominating set of $\langle V_{l-1,\pi_i^*}\rangle$ and contains a perfect matching. Then $S - \{\pi_i^*, \pi_l\}$ is a PD_{l-1,π_i^*}, and thus S is a $PD_{\pi_i^*}$. Hence, $|S| = |PD_{l-1,\pi_i^*}| + 2$. Note that $|PD_{\pi_i^*}| \le |PD_{l-1,\pi_i^*}| + 2 = |S|$, it follows that $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

Suppose $S \cap (\{\pi_{l+1}, \ldots, \pi_i\} - \{\pi_i^*\}) \ne \emptyset$. Choosing a vertex $\pi_{k_0} \in S$ $(l < k_0 < i)$, there exists $\pi_{k_0'}$ such that $(\pi_{k_0}, \pi_{k_0'}) \in M$. If $k_0' < l$, then $\pi_{k_0'} > \pi_{k_0} > \pi_{l-1}^*$, and so $(\pi_{k_0'}, \pi_{l-1}^*) \in E$. We claim that all the vertices in $N[\pi_{k_0}]$ are dominated by π_{l-1}^*, π_i^* and π_l^*. Indeed, for any $\pi_t \in N[\pi_{k_0}]$, if $t < l$, then $\pi_t > \pi_{k_0} > \pi_{l-1}^*$, so $(\pi_t, \pi_{l-1}^*) \in E$; if $l \le t \le k_0$, then $\pi_t \le \pi_l$ or $\pi_t > \pi_l > \pi_i^*$, so $\pi_t = \pi_l$, $(\pi_t, \pi_l) \in E$ or $(\pi_t, \pi_i^*) \in E$; if $t > k_0$, then $\pi_t < \pi_{k_0} < \pi_l$, so $(\pi_t, \pi_l) \in E$. The claim follows. Let $S_1 = S \cup \{\pi_{l-1}^*\} - \{\pi_{k_0}\}$. Then S_1 is a dominating set of

$\langle V_{i,j}\rangle$ and $\langle S_1\rangle$ has a perfect matching in $\langle V_i^*\rangle$ by pairing $\pi_{k'_0}$ and π^*_{l-1} and removing the edge $(\pi_{k_0}, \pi_{k'_0})$. We obtain a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \geq \max(S)$ such that $\pi^*_{l-1} \in S_1$. If $k'_0 > l$, then there exists π_{k_1} $(k_1 < l)$ such that $(\pi_{k_1}, \pi_{k'_0}) \in E$ or $(\pi_{k_1}, \pi_{k_0}) \in E$. Otherwise, since all the vertices in $\{\pi_l, \ldots, \pi_i\}$ are dominated by π_l and π_i^*, $S - \{\pi_{k_0}, \pi_{k'_0}\}$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S - \{\pi_{k_0}, \pi_{k'_0}\}\rangle$ has a perfect matching in $\langle V_i^*\rangle$ by removing $(\pi_{k_0}, \pi_{k'_0})$, contradicting the minimality of S. Hence, $\pi_{k_1} > \pi_{k_0} > \pi^*_{l-1}$ or $\pi_{k_1} > \pi_{k'_0} > \pi^*_{l-1}$. This means that $(\pi_{k_1}, \pi^*_{l-1}) \in E$. Let $S_1 = S \cup \{\pi_{k_1}, \pi^*_{l-1}\} - \{\pi_{k_0}, \pi_{k'_0}\}$. Note that all the vertices in $N(\{\pi_{k_0}, \pi_{k'_0}\})$ are dominated by π_l, π_i^* and π^*_{l-1}, so S_1 is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S_1\rangle$ has a perfect matching in $\langle V_i^*\rangle$ by pairing π_{k_1}, π^*_{l-1}, and removing the edge $(\pi_{k_0}, \pi_{k'_0})$. We again obtain a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \geq \max(S)$ such that $\pi^*_{l-1} \in S_1$. As before, by adding to S_1 the vertices in $\{\pi_1, \ldots, \pi_{l-1}\}$ and removing all the vertices of S_1 in $\{\pi_l, \ldots, \pi_i\} - \{\pi^*_{l-1}, \pi_i^*\}$, we can obtain a set $S^* \in X_1$ satisfying the conditions (i)–(iii) in Case 2 of Lemma 8 and S^* is a $PD_{\pi_i^*}$ with $\max(PD_{\pi_i^*}) = \max(S^*) \geq \max(S)$. Hence, $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$. □

By Lemmas 8–10, we obtain the following result.

Lemma 11. *For any integers i, j, if $1 < i \leq n$ and $1 \leq j \leq n$, $\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.*

Lemma 12. *For any integers i and j, $1 < i \leq n$ and $\pi_i \leq j \leq n$, if $\max(V_i) = \pi_i$, then $X_3 = \emptyset$.*

Proof. Suppose to the contrary that $X_3 \neq \emptyset$. Let $S \in X_3$. Then $\pi_i, \pi_i^* \notin S$ and S $(\subset V_i^*)$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S\rangle$ has a perfect matching in $\langle V_i^*\rangle$. Since $\pi_i \leq j \leq n$, $\pi_i \in V_{i,j}$, so π_i is dominated by a vertex π_l $(l < i)$ in S. Then $(\pi_i, \pi_l) \in E$, i.e., $(\pi_i - \pi_l)(i - l) < 0$. This implies that $\pi_l > \pi_i$, contradicting the assumption of $\max(V_i) = \pi_i$. □

Lemma 13. *For any integers i and j, $1 < i \leq n$ and $\pi_i \leq j \leq n$, if $\max(PD_{i-1,j}) < \pi_i$, then $\mathrm{Min}(X_3 \cup \{PD_{\max}\}) = PD_{\max}$.*

Proof. If $\max(V_i) = \pi_i$, by Lemma 12, $X_3 = \emptyset$. The result follows. So we may assume that $\max(V_i) \neq \pi_i$. Let Z denote the set $\{S : S \subseteq V^*_{i-1}$ and S is a dominating set of $\langle V_{i-1,j}\rangle$ and $\langle S\rangle$ has a perfect matching in $\langle V^*_{i-1}\rangle\}$. Let A be any set of X_3. Since $\pi_i \notin A$ and $\pi_i^* \notin A$, $A \subseteq V^*_{i-1}$. By Lemma 2, we have $V_{i-1,j} \subseteq V_{i,j}$, so $A \in Z$. Since $\pi_i \leq j$, $\pi_i \in V_{i,j}$, $\max(A) > \pi_i$. Thus $\max(A) > \pi_i > \max(PD_{i-1,j})$. Note that $PD_{i-1,j} = \mathrm{Min}(Z)$ and, by our definition, $\max(PD_{i-1,j})$ is as large as possible. Then it must be the case that $|A| > |PD_{i-1,j}|$. Hence, $|A| \geq |PD_{i-1,j}| + 2 = |PD_{i-1,j} \cup \{\max(V_i), \pi_i\}|$. Furthermore, $\max(A) \leq \max(V_i) = \max(PD_{i-1,j} \cup \{\max(V_i), \pi_i\})$. Therefore, $\mathrm{Min}(X_3 \cup PD_{\max}) = PD_{\max}$. □

Lemma 14. *For any integers i and j, if $1 < i \leq n$ and $1 \leq j \leq n$, then $\mathrm{Min}(X_3 \cup \{PD_{i-1,j}\}) = PD_{i-1,j}$.*

Proof. Define Z as in Lemma 13. Let A be any set of X_3. As in the proof of Lemma 13, we can verify that $A \in Z$. Note that $PD_{i-1,j} = \mathrm{Min}(Z)$. So $\mathrm{Min}(X_3 \cup \{PD_{i-1,j}\}) = PD_{i-1,j}$. □

Lemma 15. *For any integers i and j, if $1 < i \leq n$ and $1 \leq j \leq n$, then $\mathrm{Min}\{X_1 \cup X_2\} = \mathrm{Min}\{X_1\}$.*

Proof. Let $S_1 = \mathrm{Min}\{X_2\}$. According to the definition of X_2, $\pi_i^* \notin X_2$, $\pi_i \in X_2$ and $\langle S_1\rangle$ has a perfect matching M. So there exists a vertex $\pi_l \in X_2$ $(l < i)$ such that $(\pi_i, \pi_l) \in M$. Then $(\pi_l - \pi_i)(l - i) < 0$, and thus $\pi_l > \pi_i$. Hence

$$\pi_i^* < \pi_i < \pi_l \quad \text{and} \quad l < i < \pi^-(\pi_i^*). \tag{1}$$

This means that $(\pi_i^* - \pi_l)(\pi^-(\pi_i^*) - l) < 0$, i.e., $(\pi_l, \pi_i^*) \in E$. Let $S_2 = (S_1 - \{\pi_i\}) \cup \{\pi_i^*\}$. From (1) and Lemma 3, it follows that $S_2 \subseteq V_i^*$ is a dominating set of $\langle V_{i,j}\rangle$ and $\langle S_2\rangle$ has a perfect matching by pairing π_l and π_i^*. So $S_2 \in X_1$, $|S_2| = |S_1|$ and $\max(S_2) \geq \max(S_1)$. Consequently, $\mathrm{Min}\{X_1 \cup X_2\} = \mathrm{Min}\{\mathrm{Min}(X_1), \mathrm{Min}(X_2)\} = \mathrm{Min}\{\mathrm{Min}(X_1), S_1\} = \mathrm{Min}(X_1)$. □

In the following, we present the recursive formula of our dynamic programming.

Theorem 16. *For any integers i, j, if $1 < i \leq n$ and $1 \leq j \leq n$, then the following recursive formula correctly computes $PD_{i,j}$,*

$$PD_{i,j} = \begin{cases} \mathrm{Min}(\{PD_{\pi_i^*}, PD_{\max}\}) & \text{if } j \geq \pi_i \text{ and } \max(PD_{i-1,j}) < \pi_i, \\ \mathrm{Min}(\{PD_{\pi_i^*}, PD_{i-1,j}\}) & \text{otherwise.} \end{cases}$$

Proof. According to our definitions, $X = X_1 \cup X_2 \cup X_3$. By Lemmas 5 and 6, we have $PD_{\pi_i^*} \in X_1 \subseteq X$, $PD_{\max} \in X$. To complete our proof, we distinguish the following two cases.

Case 1. Suppose that $j \geq \pi_i$ and $\max(PD_{i,j}) < \pi_i$. If $\max(V_i) = \pi_i$, then, by Lemmas 11, 12 and 15, we have

$$\begin{aligned}\mathrm{Min}(X) &= \mathrm{Min}(X_1 \cup X_2 \cup \{PD_{\pi_i^*}, PD_{\max}\})\\ &= \mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}, PD_{\max}\})\\ &= \mathrm{Min}(\{\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}), PD_{\max}\})\\ &= \mathrm{Min}(\{PD_{\pi_i^*}, PD_{\max}\}).\end{aligned}$$

If $\max(V_i) \neq \pi_i$, then, by Lemmas 11, 13 and 15, we have

$$\begin{aligned}\mathrm{Min}(X) &= \mathrm{Min}(X \cup \{PD_{\pi_i^*}, PD_{\max}\})\\ &= \mathrm{Min}(X_1 \cup X_2 \cup X_3 \cup \{PD_{\pi_i^*}, PD_{\max}\})\\ &= \mathrm{Min}(X_1 \cup X_3 \cup \{PD_{\pi_i^*}, PD_{\max}\})\\ &= \mathrm{Min}(\{\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}), \mathrm{Min}(X_3 \cup \{PD_{\max}\})\})\\ &= \mathrm{Min}(\{PD_{\pi_i^*}, PD_{\max}\}).\end{aligned}$$

Case 2. Suppose that $j < \pi_i$ or $\max(PD_{i-1,j}) \geq \pi_i$. We first show that $PD_{i-1,j} \in X$. If $j < \pi_i$, then $V_{i,j} = V_{i-1,j}$, so $PD_{i-1,j} \in X$. If $\max(PD_{i,j}) \geq \pi_i$, then π_i is dominated by $PD_{i-1,j}$, so $PD_{i-1,j} \in X$. Note that $PD_{i-1,j} \subset PD_{\max}$. From Lemmas 11, 14 and 15, it follows that

$$\begin{aligned}\mathrm{Min}(X) &= \mathrm{Min}(X \cup \{PD_{\pi_i^*}, PD_{i-1,j}\})\\ &= \mathrm{Min}(X_1 \cup X_2 \cup X_3 \cup \{PD_{\pi_i^*}, PD_{i-1,j}\})\\ &= \mathrm{Min}(X_1 \cup X_3 \cup \{PD_{\pi_i^*}, PD_{i-1,j}\})\\ &= \mathrm{Min}(\{\mathrm{Min}(X_1 \cup \{PD_{\pi_i^*}\}), \mathrm{Min}(X_3 \cup \{PD_{i-1,j}\})\})\\ &= \mathrm{Min}(\{PD_{\pi_i^*}, PD_{i-1,j}\}). \quad \square\end{aligned}$$

3. An algorithm for MPDS on permutation graphs

Based on the recursive formula in Section 2, we next present the algorithmic steps to solve MPDS on permutation graphs. The overall structure of our algorithm is outlined as follows:

Algorithm: Finding an MPDS on a Permutation Graph.

Input: A permutation $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$.

Output: A minimum cardinality paired-dominating set of $G[\pi]$.

Step 1. Initialize $PD_{0,j} = \emptyset$.

$$PD_{1,j} = \begin{cases}\emptyset & \text{if } j < \pi_1,\\ \{1, \pi_1\} & \text{otherwise.}\end{cases}$$

for $j = 1, 2, \ldots, n$.

Step 2. **for** $i \leftarrow 2$ **to** n **do**

Step 3. $PD_{\pi_i^*} = \mathrm{Min}\{PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} : \pi_l \in N(\pi_i^*),\ \pi_i^* \notin PD_{l-1,\pi_i^*}, l \leq i\}$

Step 4. **for** $j \leftarrow 1$ **to** n **do**

Step 5.

$$PD_{\max} = \begin{cases}PD_{i-1,j} \cup \{\pi_i, \max(V_i)\} & \text{if } \pi_i \neq \max(V_i),\\ V_i & \text{otherwise.}\end{cases}$$

Step 6.

$$PD_{i,j} = \begin{cases}\mathrm{Min}(\{PD_{\pi_i^*}, PD_{\max}\}) & \text{if } j \geq \pi_i \text{ and } \max(PD_{i-1,j}) < \pi_i,\\ \mathrm{Min}(\{PD_{\pi_i^*}, PD_{i-1,j}\}) & \text{otherwise.}\end{cases}$$

Step 7. END
Step 8. END
Step 9. Output $PD_{n,n}$.

The time complexity of the above algorithm can be analyzed as follows. The time required in Step 3 is at most $d(\pi_i^*)$. The operations of Steps 5 and 6 can be performed in constant time. The time required in the loop from Step 4 to Step 7 is at most $O(n)$. Consequently, the overall running time of the algorithm is $O(mn)$ in an amortized sense.

Theorem 17. *Given any permutation π, the algorithm finds a minimum cardinality paired-dominating set of the permutation graph $G[\pi]$.*

Example. To illustrate our algorithm, we compute the example shown in Fig. 1. as follows:

1. $PD_{0,j} = \emptyset$;
2. $PD_{\max} = V_1$, $PD_{1,1} = PD_{1,2} = \emptyset$, $PD_{1,3} = \cdots = PD_{1,7} = \{1, 3\}$;
3. $\pi_2^* = 2$, $PD_{\pi_2^*} = \{3, 2\}$, $PD_{\max} = \{1, 3\}$, $PD_{2,1} = \cdots = PD_{2,7} = \{3, 2\}$ or $\{1, 3\}$;
4. $\pi_3^* = 2$, $PD_{\pi_3^*} = \{3, 2\}$, $PD_{\max} = V_3$, $PD_{3,1} = \cdots = PD_{3,4} = \{3, 2\}$ or $\{1, 3\}$, $PD_{3,5} = \cdots = PD_{3,7} = \{3, 2\}$;
5. $\pi_4^* = 2$, $PD_{\pi_4^*} = \{3, 2\}$, $PD_{\max} = V_4$, $PD_{4,1} = \cdots = PD_{4,4} = \{3, 2\}$ or $\{1, 3\}$, $PD_{4,5} = \cdots = PD_{4,7} = \{3, 2\}$;
6. $\pi_5^* = 2$, $PD_{\pi_5^*} = \{3, 2\}$, $PD_{\max} = \{2, 3, 7, 4\}$ or $\{1, 3, 7, 4\}$, $PD_{5,1} = \cdots = PD_{5,3} = \{3, 2\}$ or $\{1, 3\}$, $PD_{5,4} = \cdots = PD_{5,7} = \{3, 2\}$;
7. $\pi_6^* = 2$, $PD_{\pi_6^*} = \{3, 2\}$, $PD_{\max} = \{1, 3, 2, 7\}$, $PD_{6,1} = \cdots = PD_{6,3} = \{3, 2\}$ or $\{1, 3\}$, $PD_{6,4} = \cdots = PD_{6,7} = \{3, 2\}$;
8. $\pi_7^* = 6$, $PD_{\pi_7^*} = \{3, 2, 7, 6\}$, $PD_{\max} = \{3, 2, 7, 6\}$ or $\{1, 3, 7, 6\}$, $PD_{7,1} = \cdots = PD_{7,3} = \{3, 2, 7, 6\}$ or $\{1, 3, 7, 6\}$, $PD_{7,4} = \cdots = PD_{7,7} = \{3, 2, 7, 6\}$.

In light of our algorithm, $PD_{7,7} = \{3, 2, 7, 6\}$ is a minimum cardinality paired-dominating set of the graph.

4. Conclusions

In this paper we presented an $O(mn)$ algorithm for finding a minimum cardinality paired-dominating set for a permutation graph with order n and size m. Our algorithm is based on a recursive formula in conjunction with applying the dynamic programming method. The idea was previously used by Chao et al. [6] for finding the minimum cardinality dominating set on permutation graphs. We speculate that the time complexity of the MPDS problem on permutation graphs can be reduced to $O(n \log n)$ and we suggest that researchers investigate such a possibility. It is also interesting to determine whether there exist some other classes of graphs in which the minimum paired-domination problem is polynomially solvable.

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